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OCT 78 C E BLAIR, J BORWEIN, R G JEROSLOW N00014-75-C-0621
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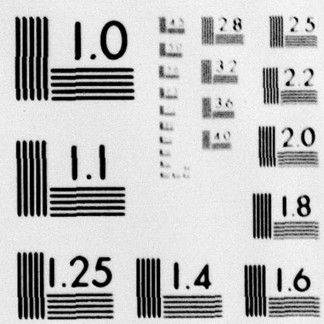

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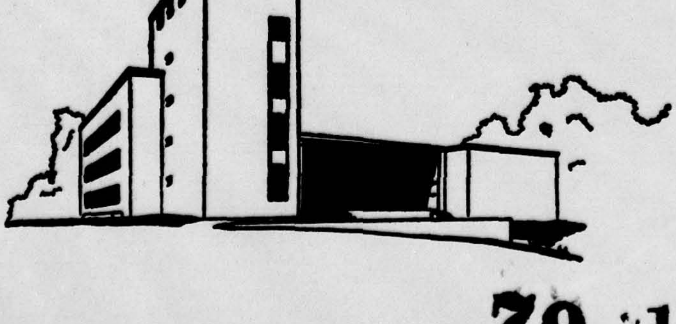

Management Sciences Research Report No. 430

CONVEX PROGRAMS AND THEIR CLOSURES

by

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October 1978

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This report was prepared as part of the activities of the Management Sciences Research Group, Carnegie-Mellon University, and has been supported by NSF grant GP 37510 X1, and under Contract N00014-75-C-0621 NR 047-048 with the U. S. Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the U. S. Government.

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Abstract

We extend the limiting lagrangean equation

limit as theta approaches 0(+) — theta — sum over h as elements of H of

$$\lim_{\theta \to 0^+} \sup_{\Lambda} \inf_{x \in K} \left\langle f_0''(x) + \theta(wx + w_1'') + \sum_{h \in H} \lambda_h f_h''(x) \right\rangle = v(P) \quad ,$$

sup over lambda — inf over x as elements of K — lambda h

and the results on affine supports from which it was deduced, to a very general setting that subsumes the previous constraint qualifications.

A simple example shows the need for some constraint qualification.

Key Words:

(1) Nonlinear programming

(2) Lagrangean

(3) Convexity

CONVEX PROGRAMS AND THEIR CLOSURES

by

C. E. Blair,[1] J. Borwein,[2] and R. G. Jeroslow[3]

For a convex function $f: D \to R$ ($D \subseteq R^n$, D convex) the closure $cl(f): cl(D) \to R \cup \{+\infty\}$ is defined by

$$\text{(1)} \qquad cl(f)(y) = \sup\{h(y) \mid h \text{ linear affine, } h(x) \le f(x) \text{ for all } x \in D\}$$

where $cl(D)$ is the closure of the convex set D.

It is well known that: (i) $cl(f)(x) \le f(x)$ for all $x \in D$; (ii) $cl(f)$ is convex; (iii) $cl(f)(x) = f(x)$ for all $x \in relint(D)$, where $relint(D)$ denotes the relative interior of the convex set D.

For a convex optimization problem (with possibly infinitely many constraints)

$$\text{(P)} \qquad \begin{array}{c} \inf f_0(x) \\ \text{subject to } f_h(x) \le 0 \text{ for } h \in H \\ \text{and } x \in K \end{array}$$

with optimal value denoted $v(P)$, the closure is

$$\text{(P')} \qquad \begin{array}{c} \inf cl(f_0)(x) \\ \text{subject to } cl(f_h)(x) \le 0 \text{ for } h \in H \\ \text{and } x \in cl(K) \end{array}$$

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with optimal value denoted $v(P')$.

We assume throughout that (P) is consistent.

Duffin [1] and Jeroslow [2] show that when (P) and (P′) have the same optimal value, a "limiting Lagrangean" exists, in the sense that (using the homotopy form of the limiting Lagrangean of [2, equation (50)])

$$\text{(2)} \qquad \lim_{\theta \to 0+} \sup_{\Lambda} \inf_{x \in K} \{f_0(x) + \theta(wx + w_1) + \sum_{h \in H} \lambda_h f_h(x)\} = v(P)$$

for $w \in R^n$ and $w_1 \in R$ suitably chosen, where Λ denotes that space of vectors $(\lambda_h | h \in H)$ which are nonnegative and only finitely non-zero. Moreover, from the value equality $v(P) = v(P')$ also follows "fine detail" from which (2) is deduced, as e.g., [2, Theorem 3] and [2, Corollary 3].

It is also established in [2] that $v(P) = v(P')$ holds in many instances in which the usual constraint qualifications, such as the existence of Slater points, may fail to hold even for $|H|$ finite. This is because the limiting lagrangean (2) is not related to issues of linear affine, or even rather more general, supports to the perturbation function of (P). This aspect of the limiting lagrangean was already present in the first limiting lagrangean, due to R. J. Duffin [1].

The purpose of this note, is to extend the validity of the limiting Lagrangean (and Theorem 3 and Corollary 3 of [2]) to a rather broad setting that is associated with the ordinary lagrangean in the case of $|H|$ finite. We show, in this setting, that the limiting lagrangean holds again under weaker hypotheses than the ordinary lagrangean, even for $|H|$ finite; and our result also treats $|H|$ infinite.

Let D_h be the domain of definition of f_h. [2] showed that $v(P) = v(P')$ if $\mathrm{relint}(K) \subseteq \mathrm{relint}(D_h)$ for all $h \in \{0\} \cup H$ and there was an $x_0 \in \mathrm{relint}(K)$ such that $f_h(x_0) \leq 0$ for all $h \in H$. These latter hypotheses were called (CQ) in [2].

In this note we show:

<u>THEOREM</u>: Let H' denote those indices $h \in \{0\} \cup H$ such that f_h is not closed.

$v(P) = v(P')$ if there is an x_0 satisfying this constraint qualification:

$$(3) \qquad x_0 \in \mathrm{relint}(K) \cap \bigcap_{h \in H'} \mathrm{relint}(D_h) \text{ and } f_h(x_0) \leq 0 \text{ for } h \in H \ .$$

(The intersection over an empty set is defined to be R^n).

<u>PROOF</u>: If x is feasible for (P) it is also feasible for (P′) because $\mathrm{cl}(f_h)(x) \leq f_h(x)$. Since $\mathrm{cl}(f_0)(x) \leq f_0(x), v(P') \leq v(P)$.

To show that $v(P') \geq v(P)$, let x be any feasible point of (P′). For $0 < \lambda < 1$ if $y = \lambda x + (1 - \lambda)x_0$, $y \in K$ and $y \in \mathrm{relint}(D_h)$ for all $h \in H'$, by the Accessibility Lemma [5, 3.2.11]. By (iii), $\mathrm{cl}(f_h)(y) = f_h(y)$ for all $h \in H'$; therefore $\mathrm{cl}(f_h)(y) = f_h(y)$ for all $h \in H \cup \{0\}$.

Since $\mathrm{cl}(f_h)(x_0) \leq f_h(x_0) \leq 0$ and $\mathrm{cl}(f_h)(x) \leq 0$ for $h \in H$, one easily shows (by considering f_h and $\mathrm{cl}(f_h)$ on $[x,x_0]$) that $f_h(y) = \mathrm{cl}(f_h)(y) \leq 0$ for $h \in H$. So y is a feasible point for (P).

By semi-continuity $\lim_{\lambda \to 1} \mathrm{cl}(f_0)(y) \leq \mathrm{cl}(f_0)(x)$. But since $\mathrm{cl}(f_0)(y) = f_0(y)$ for all $\lambda < 1$ this implies $v(P) \leq \mathrm{cl}(f_0)(x)$.

Since x was arbitrary, this shows $v(P) \leq v(P')$. Hence $v(P) = v(P')$, as desired.

Q.E.D.

<u>REMARK</u>: The same proof shows that, if K is closed, one obtains $v(P) = v(P')$ from:

(4) $$x_0 \in K \cap \bigcap_{h \in H'} \text{relint } (D_h)$$

$$\text{and } f_h(x_0) \le 0 \text{ for } h \in H.$$

In one of the constraint qualifications of [2], it is assumed that $H' = \phi$ and K is closed, in which case (4) becomes that constraint qualification (CQ)$'$. Trivially, (4) implies also the constraint qualification (CQ) of [2].

<u>COROLLARY</u>: Suppose that (P) has at least two different feasible points, and that none of the sets K or D_h for $h \in H'$ contains any line segments in $K \backslash \text{relint}(K)$ or $D_h \backslash \text{relint}(D_h)$ (where H' is as defined in the theorem).

Then $v(P) = v(P')$.

<u>PROOF</u>: Let $x_a \neq x_b$ both be feasible in (P). Since $x_a, x_b \in K$ and K contains no line segment in $K \backslash \text{relint}(K)$, $x_0 = (x_a + x_b)/2 \in \text{relint}(K)$. Similarly, $x_0 \in \text{relint}(D_h)$ for $h \in H'$. Trivially, $f_h(x_0) \le 0$ for $h \in H$.

The result now follows from the theorem.

Q.E.D.

Some constraint qualification is needed to insure that $v(P) = v(P')$, even for $|H|$ finite. For example, consider this instance of a convex program:

$$
(5) \qquad \begin{aligned} &\inf x_1 \\ \text{subject to} \quad & x_2 \le 0 \\ & f_2(x_1,x_2) \le 0 \\ & (x_1,x_2) \in K \end{aligned}
$$

where $K = \{(x_1,x_2) | 0 \le x_1 \le 1 \text{ and } x_2 \ge 0\}$ and

$$
(6) \qquad f_2(x_1,x_2) = \begin{cases} 0 \quad , & 0 \le x_1 \le 1 \text{ and } x_2 > 0; \\ 1 - x_1, & 0 \le x_1 \le 1 \text{ and } x_2 = 0; \\ +\infty \quad , & \text{otherwise}. \end{cases}
$$

Here $v(P) = 1$ and $v(P') = 0$, since $\mathrm{cl}(f_2)(x_1,x_2) \equiv 0$ if $0 \le x_1 \le 1$ and $x_2 \ge 0$. In this example, $H' = \{2\}$, as f_2 is not continuous on line segments that begin in the interior of K and end in the boundary segment $\{(x_1,x_2) | x_2 = 0 \text{ and } 0 \le x_1 < 1\}$. Here also $x_2 \le 0$ and $(x_1,x_2) \in K$ implies $x_2 = 0$, so $(x_1,x_2) \notin \mathrm{relint}(D_2)$; hence (3) fails.

August 2, 1978
Revised September 14, 1978

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Report No. 430	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) CONVEX PROGRAMS AND THEIR CLOSURES.		5. TYPE OF REPORT & PERIOD COVERED Technical Report October 1978
		6. PERFORMING ORG. REPORT NUMBER (14) MSRR-430
7. AUTHOR(s) (10) C. E. Blair, J. Borwein R. G. Jeroslow		8. CONTRACT OR GRANT NUMBER(s) [redacted] (15) N00014-75-C-0621 NSF-GP37571A
9. PERFORMING ORGANIZATION NAME AND ADDRESS Graduate School of Industrial Administration Carnegie-Mellon University Pittsburgh, Pennsylvania 15213		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 047-048
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE (11) Oct[redacted]78
		13. NUMBER OF PAGES 6
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited. (12) 10

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

(9) Management sciences research rept.,

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Nonlinear programming, Lagrangean, Convexity

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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S/N 0102-014-6601

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